D Van Kampen's Theorem

let G_1 and G_2 be two groups

a word in $G_1 \cup G_2$ is a finite sequence

$$x = (x_1, x_2, \dots x_n) \quad \text{some } n$$

where each x_i is an element in $G_1 \cup G_2$

define an equivalence relation on the set of words generated by

(1) replace a, b in a sequence with $a \cdot b$ if a, b are in same G_i
(and the reverse of this)

(2) if e_{G_i} is in a sequence remove it (here e_{G_i} is the identity in G_i)
(and the reverse of this)

exercise: Show this is an equivalence rel$^{\underline{n}}$

denote the equivalence class of x by $[x]$

call a word $x = (x_1, x_2, \dots, x_n)$ reduced if

$x_i \neq e_{G_i}$ (the identity in either group)

x_i, x_{i+1} from different groups

exercise: each equivalence class $[x]$ contains a unique reduce word.

the free product of G_1 and G_2 is the group $G_1 * G_2$ of all equivalence classes of words in $G_1 \cup G_2$

multiplication is $[x_1, \dots, x_m] \cdot [y_1, \dots y_n] = [x_1, \dots x_m, y_1, \dots y_n]$

and let e = empty word

note: $ex = xe = x \quad \forall x$

$x^{-1} = (x_m^{-1}, \dots, x_1^{-1})$

exercise: Check multiplication is associative (induct on length)

Remark: we could define $G_1 * G_2$ to be the collection of reduced words in $G_1 \cup G_2$ with multiplication

$$(x_1, \dots, x_m) \cdot (y_1, \dots, y_n) = \begin{cases} \text{unique reduced word} \\ \text{in } [x_1, \dots, x_m, y_1, \dots y_n] \end{cases}$$

Property of free products

let $j_i : G_i \to G_1 * G_2$ be the obvious inclusion $i = 1,2$

given any homomorphisms $\phi_i : G_i \to H$ where H any group

$\exists !$ homomorphism $\phi : G_1 * G_2 \to H$ (just apply ϕ_i to letters in word)

s.t. $\phi \circ j_i = \phi_i \quad i = 1,2$

Pictorially

$$\begin{tikzcd} & G_1 \arrow[d, "j_1"] \arrow[ddr, bend left, "\phi_1"] & \\ G_2 \arrow[r, "j_2"] \arrow[drr, bend right, "\phi_2"'] & G_1 * G_2 \arrow[dr, dashed, "\phi"] & \\ & & H \end{tikzcd}$$

exercise: Show property above defines the free product

i.e. if D is any other group satisfying property then $D \cong G_1 * G_2$

the above property called a universal property *example:*

$$\underset{\{x^n\}}{\overset{\shortparallel}{\mathbb{Z}}} * \underset{\{y^n\}}{\overset{\shortparallel}{\mathbb{Z}}} = \{x^{n_1}y^{m_1}\cdots x^{n_k}y^{m_k},\ x^{n_1}y^{m_1}\cdots x^{n_k},\ y^{m_1}x^{n_1}\cdots x^{n_k},\ y^{m_1}x^{n_1}\cdots y^{n_k},\ e\}$$

complete list of elts $n_1, m_1 \neq 0$

this is called the free group on 2 generators and denoted F_2

in general the free group on n generators is

$$F_n = F_{n-1} * \mathbb{Z} = \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{n\text{-times}} \qquad (\text{also have } F_\infty)$$

$= \{$all words in n-letter alphabet, and inverses$\}$

note: a homomorphism $f: \mathbb{Z} \to G$ is uniquely specified by $f(1)$ and for any choice $g \in G$, $\exists!$ homomorphism sending 1 to g.

From above property we see we get a unique homomorphism $F_n \to H$ once we specify where the n generators go

a group presentation is a group

$$\langle X \mid R \rangle$$

where X is some set

and R is a collection of words in the letters $X \cup X^{-1}$

X^{-1} is a copy of X where $x \in X$ is denoted x^{-1} in X^{-1}

cardinality of X

let F_n = free group on n generators where $n = |X|$

so we can think of F_n as words in $X \cup X^{-1}$

$\therefore$ R is a collection of elements in F_n

let $\langle R \rangle$ be the smallest normal subgroup of F_n containing R

the group that $\langle X \mid R \rangle$ is $F_n / \langle R \rangle$

intuitively: $\langle x_1, \dots, x_n \mid r_1 \dots r_m \rangle$ is the group of all words in x_i and x_i^{-1} where if you see r_i you can remove it (you can also insert it anywhere)

examples:

1) $\langle g \mid g^n \rangle$ this is all words in g and g^{-1}

$$\dots g^{-2}, g^{-1}, e, g, g^2, g^3, \dots, g^n, \dots$$

but $g^n = e$ so $g^{n+1} = g^n g = g$

and $g^{-1} = g^n g^{-1} = g^{n-1}$

so every element is of the form g^k $\quad 0 \le k < n$

exercise: $\langle g \mid g^n \rangle \to \mathbb{Z}_n$ ← integers modulo n

$g^k \longmapsto [k]$ is an isomorphism

so $\langle g \mid g^n \rangle$ is a presentation of $\mathbb{Z}_n$

2) a presentation of $\mathbb{Z}$ is $\langle g \mid \; \rangle$

lemma 15: $\boxed{\text{every group } G \text{ has a presentation}}$

Proof: given G, let X be a subset of G that generates G

let $n = |X|$ (could be ∞)

$\exists!\, \phi : F_n \to G$

generator to elt X

let $N = \ker \phi \trianglelefteq F_n$ (normal subgroup)

1st isomorphism theorem says

$G \cong F_n / N$

let R be elements of N that generat N

so $G \cong \langle X \mid R \rangle$

exercises:

1) If $G = \langle g_1, \dots, g_n \mid r_1 \dots r_m \rangle$, then for any group H and

any map $\phi : \{g_1, \dots g_n\} \to H$ satisfying $\phi(r_i) = e_H$

$\exists!$ homomorphism $\Phi : G \to H$ s.t. $\Phi(g_i) = \phi(g_i)$

2) if $G_1 = \langle g_1, \dots g_n \mid r_1 \dots r_m \rangle$

$G_2 = \langle h_1 \dots h_k \mid s_1, \dots s_\ell \rangle$

then $G_1 * G_2 \cong \langle g_1, \dots g_n, h_1 \dots h_k \mid r_1 \dots r_m, s_1, \dots s_\ell \rangle$

given groups G_1, G_2 and K and homomorphisms

$$\psi_i : K \to G_i \qquad i=1,2$$

then the free product with amalgamation is

$$G_1 *_K G_2 = G_1 * G_2 \Big/ \langle \psi_1(k)(\psi_2(k))^{-1} \rangle_{k\in K}$$

where $\langle \psi_1(k)(\psi_2(k))^{-1} \rangle_{k\in K}$ is the smallest normal subgroup of $G_1 * G_2$ containing the set $\{\psi_1(k)(\psi_2(k))^{-1}\}_{k\in K}$

the idea here is that we have words in the elts of G_i but when you see $\psi_1(k)$ in a word you can replace it with $\psi_2(k)$

in terms of presentations, if $G_1 = \langle g_1 \dots g_n \mid r_1 \dots r_m \rangle$

$$G_2 = \langle g_1' \dots g_{n'}' \mid r_1' \dots r_{m'}' \rangle$$

$$K = \langle h_1 \dots h_k \mid s_1 \dots s_l \rangle$$

then

$$G_1 *_K G_2 \cong \langle g_1 \dots g_n, g_1' \dots g_{n'}' \mid r_1 \dots r_m, r_1' \dots r_{m'}', \psi_1(h_1)(\psi_2(h_1))^{-1} \dots \psi_1(h_k)(\psi_2(h_k))^{-1} \rangle$$

exercises:

1) prove presentation above is correct

2) let $j_i : G_i \to G_1 * G_2$ be the natural inclusion maps and $\bar{J}_i : G_i \to G_1 *_K G_2$ the induced maps

given homomorphisms $\phi_i : G_i \to H$ (H any group)

such that $\phi_1 \circ \psi_1(k) = \phi_2 \circ \psi_2(k) \quad \forall \; k \in K$

then $\exists!$ homomorphism $\Phi : G_1 *_K G_2 \to H$ s.t. $\Phi \circ \bar{J}_i = \phi_i$

Pictorally:

$$\begin{tikzcd}
 & G_1 \arrow[dr, "\bar{J}_1"] \arrow[rrd, bend left, "\phi_1"] & & \\
K \arrow[ur, "\psi_1"] \arrow[dr, "\psi_2"'] & & G_1 *_K G_2 \arrow[r, dashed, "\Phi"] & H \\
 & G_2 \arrow[ur, "\bar{J}_2"'] \arrow[rru, bend right, "\phi_2"'] & &
\end{tikzcd}$$

this is the universal property for free products with amalgamation

Thm 16 (Seifert-Van Kampen):

let X be a path connected space with base point x_0

suppose $X = A \cup B$ where

A, B and $A \cap B$ are open, path connected sets and

$x_0 \in A \cap B$

let $\psi_A : \pi_1(A\cap B, x_0) \to \pi_1(A, x_0)$ and

$\psi_B : \pi_1(A \cap B, x_0) \to \pi_1(B, x_0)$

be the maps induced by inclusion

$$A \cap B \begin{smallmatrix} \subset A \\ \subset B \end{smallmatrix}$$

Then

$$\pi_1(X, x_0) \cong \pi_1(A, x_0) *_{\pi_1(A\cap B, x_0)} \pi_1(B, x_0)$$

Remark: More generally, if $\{A_\alpha\}$ a collection of path-connected open sets in X, $x_0 \in A_\alpha$, $A_\alpha \cap A_\beta$, $A_\alpha \cap A_\beta \cap A_\gamma$ path connected $\forall\ \alpha, \beta, \gamma$ then

$$\Phi : \ast_\alpha \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0) \quad \text{surjective with } \ker \Phi$$

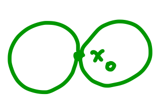
the smallest normal subgroup N containing $(\iota_{\alpha\beta})_*(g)\left((\iota_{\beta\alpha})_*(g)\right)^{-1}$

where $\iota_{\alpha\beta} : A_\alpha \cap A_\beta \to A_\alpha$ is inclusion

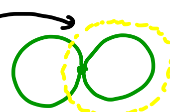
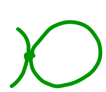
we can now compute lots of fundamental groups

example: 1)

let $W_2 =$ "wedge of two circles" $= S^1 \vee S^1$

$=$ 

think of as CW complex or subset of $\mathbb{R}^2$

let A  $=$ 

and similarly $B =$ 

so $W_2 = A \cup B$

$x_0 \in A \cap B =$  path connected

exercise: 1) $A \cap B \simeq \{x_0\}$

2) $A \simeq S^1 \simeq B$

so $\pi_1(A, x_0) \cong \mathbb{Z} \cong \langle g_1 | \ \rangle$

$\pi_1(B, x_0) \cong \mathbb{Z} \cong \langle g_2 | \ \rangle$

$\pi_1(A \cap B, x_0) \cong \{e\}$

$\psi_A : \pi_1(A \cap B, x_0) \to \pi_1(A, x_0)$ similarly for ψ_B
$e \longmapsto e$

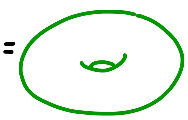
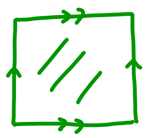
so

$$\pi_1(W_2, x_0) \cong \pi_1(A, x_0) *_{\pi_1(A\cap B, x_0)} \pi_1(B, x_0)$$
$$\cong \langle g_1 | \ \rangle *_{\{e\}} \langle g_2 | \ \rangle$$
$$\cong \langle g_1, g_2 | \ \rangle \cong F_2$$ the free group on 2 generators

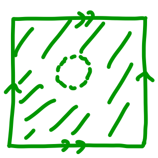
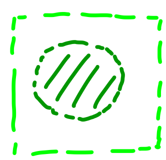
<u>exercise:</u> 1) if $W_n = W_{n-1} \vee S^1$  wedge of n-circles

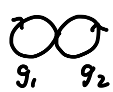
then $\pi_1(W_n, x_0) \cong F_n$

2) if X is any connected graph, then $\pi_1(X) \cong F_n$ for some n

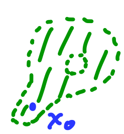
2) $T^2 =$  $\cong$ 

(we know $T^2 = S^1 \times S^1$ so $\pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$, but we now compute via Van Kampen)

let $A =$  and $B =$  $\cong$ 

<u>exercise:</u> 1) $A \simeq$  $= W_2$

2) $B \simeq \{pt\}$

3) $A \cap B =$  $\cong$  $\simeq$  S^1

so $\pi_1(A, x_0) \cong \langle g_1, g_2 | \ \rangle$

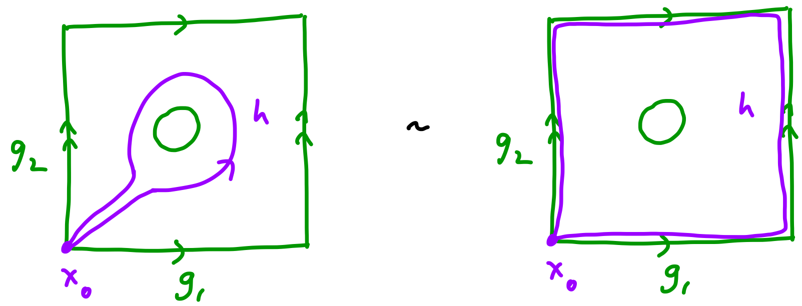
$\pi_1(B, x_0) = \{e\}$

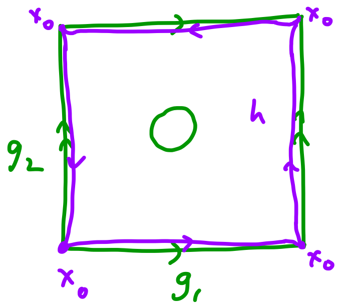
$\pi_1(A \cap B, x_0) = \langle h | \ \rangle$

<u>note:</u> $\psi_B : \pi_1(A \cap B, x_0) \to \pi_1(B, x_0)$ trivial map

what about ψ_A ?

Claim $\psi_A(h) = g_1 g_2 g_1^{-1} g_2^{-1}$





$$h \sim g_1 * g_2 * \bar{g}_1 * \bar{g}_2$$

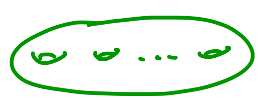
so $\pi_1(T^2, x_0) \cong \pi_1(A, x_0) *_{\pi_1(A \cap B, x_0)} \pi_1(B, x_0)$

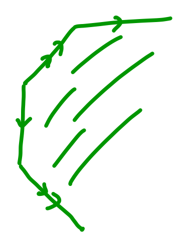
$\cong \langle g_1, g_2 \mid \ \rangle *_{\langle h \mid \ \rangle} \{e\}$

$\cong \langle g_1, g_2 \mid g_1 g_2 g_1^{-1} g_2^{-1} \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$

exercise

exercise: 1) if Σ_n is the surface of genus n

 $\cong$ 4n-gon w/ identifications



then $\pi_1(\Sigma_g, x_0) = \langle g_1, g_2 \dots g_{2n} \mid \prod_{i=1}^{n} [g_{2i-1}, g_{2i}] \rangle$

where $[h, k] = h k h^{-1} k^{-1}$

"commutator"

2) this is non abelian for $n>1$

(quotient out even g_n's)

3) $\Sigma_n \cong \Sigma_m \Leftrightarrow \Sigma_n \simeq \Sigma_m \Leftrightarrow n=m$

Th^{m} 17:

let X be path connected

$f: \partial D^n \to X$ be continuous

$x_0 \in \partial D^n$

Set $Y = X \cup_f D^2 = (X \amalg D^2) / _{x \in \partial D^2 \sim f(x) \in X}$

Then

$$\pi_1(Y, x_0) \cong \begin{cases} \pi_1(X, f(x_0)) * \mathbb{Z} & n=1 \quad ⊛ \\ \pi_1(X, f(x_0)) / \langle r \rangle & n=2 \quad r = f_*(g) \text{ and } g \text{ generates } \pi_1(\partial D^2, x_0) \\ \pi_1(X, f(x_0)) & n \geq 3 \end{cases}$$

⊛ need X to have a point $f(x_0) \in X$ with open nbhd $U \simeq \{f(x_0)\}$

Remarks: 1) So attaching 1-cell adds generator to π_1 (usually)

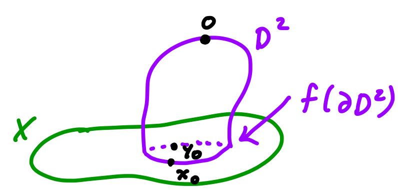
" " 2-cell adds a relation to π_1

" " $n \geq 3$-cell doesn't change π_1

2) This allows us to compute π_1 of any CW complex!

$\therefore$ π_1 of any manifold and many other spaces

Proof: $\underline{n=2:}$



let $A = X \cup_f \underbrace{S^1 \times (0,1]}_{D^2 - \{0\}} \simeq X$

B = interior of $D^2 \simeq \{pt\}$

$A \cap B = (\text{int } D^2) - \{0\} \simeq S^1$ let $y_0 \in A \cap B$ that goes to $f(x_0) \in X$ under def. retraction $A \simeq X$

$$\begin{array}{ccc} \psi_A : \pi_1(A \cap B, y_0) & \longrightarrow & \pi_1(A, y_0) \\ \wr\| & & \wr\| \\ \langle g \rangle & & \pi_1(X, f(x_0)) \\ g & \longmapsto & f_*(g) \end{array}$$

$\psi_B(g) = e$

so $\pi_1(Y, x_0) \cong \pi_1(X, f(x_0)) *_{\mathbb{Z}} \{e\} \cong \pi_1(X, f(x_0)) / \langle f_*(g) \rangle$ ✓

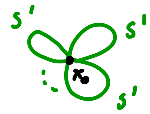
exercise: Work out cases $n \neq 2$

Why do you need ⊛ for $n=1$?

Cor 18: $\boxed{\text{let } G \text{ be any group with finite presentation then } \exists \text{ a topological space } X \text{ s.t. } \pi_1(X,x_0) \cong G}$

Proof: let $G \cong \langle g_1, \dots, g_n \mid r_1 \dots r_m \rangle$

let W_n = wedge of n circles



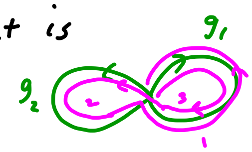
$\pi_1(W_n, x_0) \cong \langle g_1, \dots, g_n \mid \ \rangle$

for each r_i let $f_i : \partial D^2 \to W_n$ be a continuous map

s.t. $(f_i)_*(1) = r_i$

$\uparrow$ gen of $\mathbb{Z} = \pi_1(S^1)$

exercise: Why do we know such an f_i exists?

(what is  ?)

let $X = W_n \cup_{f_i} \left(\coprod_{i=1}^{m} D^2 \right)$

$\pi_1(X, x_0) \cong G$ by Th$^{\underline{m}}$ 17

Proof of Seifert-VanKampen Th$^{\underline{m}}$ 16:

let $\phi_A : \pi_1(A, x_0) \to \pi_1(X, x_0)$

$\phi_B : \pi_1(B, x_0) \to \pi_1(X, x_0)$ be the homomorphisms induced by inclusions $A \subseteq X$, $B \subseteq X$

let

$$\Phi : \pi_1(A, x_0) * \pi_1(B, x_0) \to \pi_1(X, x_0)$$

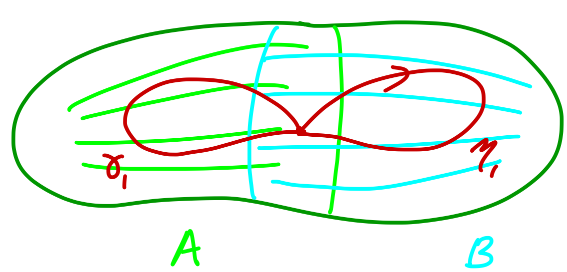
be the homomorphism induced on the free product

i.e. $\Phi([\gamma_1][\eta_1] \dots [\gamma_n][\eta_n]) = \phi_A([\gamma_1]) \cdot \phi_B([\eta_1]) \cdots \phi_B([\eta_n])$

note: if $[\gamma] \in \pi_1(A \cap B, x_0)$

then $\phi_A \circ \psi_A([\gamma]) = [\gamma] = \phi_B \circ \psi_B([\gamma])$

$\uparrow$ as loops



so $\Phi\left(\psi_A([\gamma]) \left(\psi_B([\gamma]) \right)^{-1} \right) = [\gamma] \cdot [\gamma]^{-1} = e$

so $K = \langle \psi_A([\gamma])(\psi_B([\gamma]))^{-1} \rangle_{[\gamma] \in \pi_1(A \cap B, x_0)} \subset \ker \Phi$

$\therefore$ Φ induces a homomorphism, still denoted Φ,

$$\Phi : \pi_1(A, x_0) *_{\pi_1(A \cap B, x_0)} \pi_1(B, x_0) \to \pi_1(X, x_0)$$

lemma 9 say Φ is surjective!

<u>Claim:</u> Φ injective

suppose $[\gamma_1] \in \pi_1(A, x_0)$, $[\eta_1] \in \pi_1(B, x_0)$

and $\Phi([\gamma_1][\eta_1] \cdots [\eta_n]) = [\gamma_1 * \eta_1 \cdots * \eta_n] = e$ $\circledast$

We need to show we can get from the word

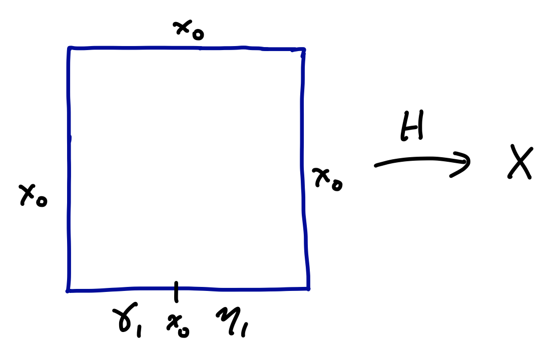
$[\gamma_1], [\eta_1], \ldots, [\eta_n]$

to the empty word by a sequence of

doesn't change word in $\pi_1(A) * \pi_1(B)$ } (1) replace a, b by $a \cdot b$ if a, b in same group (and reverse of this)

doesn't change word in $\pi_1(A) *_{\pi_1(A \cap B)} \pi_1(B)$ } (2) if we see $\psi_A(k)$ then replace it with $\psi_B(k)$ (and the converse of this)

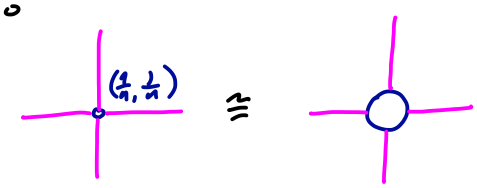
to this end $\circledast$ says we have a homotopy



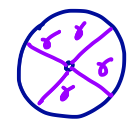
as in proof of lemma 12(b) can use Lebesgue number lemma to find n s.t. squares of side length $\frac{1}{n}$ are mapped by H into A or B (can assume # of γ_1, η_1 divides n)

exercise: can assume $H(\frac{1}{n}, \frac{1}{n}) = x_0$
(change H and γ_1, η_1 but by homotopy in A or B, respectively)

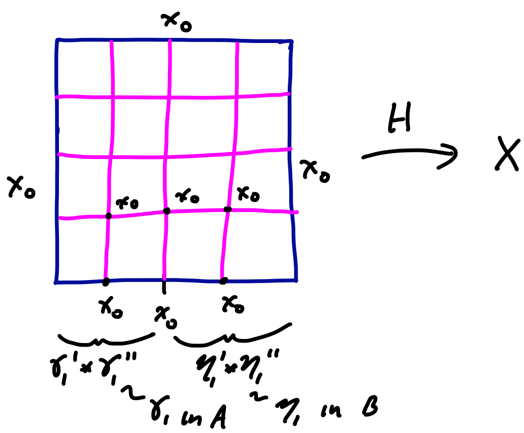
hint:



on radial lines of disk use path γ x_0 to original $H(\frac{i}{n}, \frac{j}{n})$

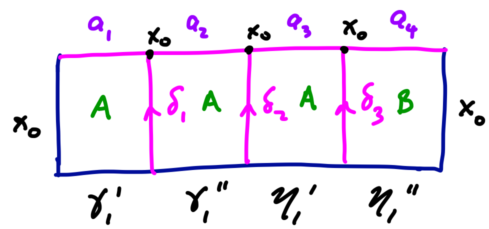


so we have



look at bottom row

e.g.



each $a_i \subset A$ or B

$\delta_i \subset A$ or B in $A \cap B$

if adjacent squares not both in same A or B

note:

$$\left.\begin{aligned}
&\gamma_1 * \eta_1' \sim \gamma_1 * \eta_1' * \delta_3 * \bar{\delta}_3 && \text{in } G\\
&\gamma_1 * \eta_1' * \delta_3 \sim a_1 * a_2 * a_3 && \text{in } G\\
&\bar{\delta}_3 * \eta_1'' \sim a_4 && \text{in } H
\end{aligned}\right\} \Rightarrow$$

$$\begin{aligned}
[\gamma_1]^G, [\eta_1]^H &= [\gamma_1]^G, ([\eta_1'] \cdot [\eta_1''])^H\\
&\overset{(1)}{=} [\gamma_1]^G, [\eta_1']^H, [\eta_1'']^H\\
&\overset{(2)}{=} [\gamma_1]^G, [\eta_1']^G, [\eta_1'']^H\\
&\overset{(1)}{=} ([\gamma_1] \cdot [\eta_1'])^G, [\eta_1'']^H\\
&= [\gamma_1 * \eta_1' * \delta_3 * \bar{\delta}_3]^G, [\eta_1'']^H\\
&= ([\gamma_1 * \eta_1' * \delta_3] \cdot [\bar{\delta}_3])^G, [\eta_1'']^H\\
&\overset{(1)}{=} [\gamma_1 * \eta_1' * \delta_3]^G, [\bar{\delta}_3]^G, [\eta_1'']^H\\
&\overset{(2)}{=} [\gamma_1 * \eta_1' * \delta_3]^G, [\bar{\delta}_3]^H, [\eta_1'']^H\\
&\overset{(1)}{=} [\gamma_1 * \eta_1' * \delta_3]^G, ([\bar{\delta}_3] \cdot [\eta_1''])^H\\
&= [\gamma_1 * \eta_1' * \delta_3]^G, [\bar{\delta}_3 * \eta_1'']^H\\
&= [a_1 * a_2 * a_3]^G, [a_4]^H
\end{aligned}$$

$G = \pi_1(A, x_0)$
$H = \pi_1(B, x_0)$

so we can go from one line $[0,1] \times \{\frac{i}{n}\}$ to next by "amalgamation relations"

The "formal proof" (maybe don't do in class)

we show how to go from the word defined by $[0,1]\times\{\frac{i}{n}\}$

to the one given by $[0,1]\times\{\frac{i+1}{n}\}$ using (1) and (2)

inductively we have a sequence of n elements $[a_1]\dots[a_n]$

where $a_j = H|_{[\frac{j}{n},\frac{j+1}{n}]\times\{\frac{i}{n}\}}$

each $[a_j] \in G = \pi_1(A,x_0)$ or $H = \pi_1(B,x_0)$

(when $i=0$, need to use (1) to break γ_1, η_1 into smaller loops)

<u>Step 1</u>: Use (2) to arrange each $[a_j]$ is in G or H according to whether the "square above" a_j is in A or B, respectively

<u>Step 2</u>: if at $\frac{j}{n}$ the "squares" change from being in A to being in B (or vice versa)

then add $\delta_j = H_{\{\frac{j}{n}\}\times[\frac{i}{n},\frac{i+1}{n}]}$ and $\bar{\delta}_j$ to $[a_{j-1}]$

i.e. $[a_{j-1}] = [a_{j-1} * \delta_j * \bar{\delta}_j]$

<u>Step 3</u>: $$[a_{j-1}*\delta_j*\bar{\delta}_j]^G, [a_j]^H \overset{(1)}{=} [a_{j-1}*\delta_j]^G, [\bar{\delta}_j]^G, [a_j]^H$$

$$\underset{(2)}{=} [a_{j-1}*\delta_j]^G, [\bar{\delta}_j]^H [a_j]^H \underset{(1)}{=} [a_{j-1}*\delta_j]^G, [\bar{\delta}_j * a_j]^H$$

<u>Step 4</u>: Use (1) to combine loops in same G or H (i.e. make reduced word)

and note each letter in reduced word is also represented by a product of paths

$a_j' = H_{[\frac{j}{n},\frac{j+1}{n}]\times\{\frac{i+1}{n}\}}$ (via homotopy H)

<u>Step 5</u>: Use (1) to breack this word into $[a_1'],\dots,[a_n']$